Technical Note: Orthogonal “Non-Interacting” Signal Splits via Walsh–Hadamard Structure

1. Intuition: two signals that *cannot see each other*

Imagine you take a signal x and split it into two parts… but not in the usual way (like low/high frequency, or positive/negative values).

Instead, you split it so that:

- The two parts are **perfectly orthogonal**
- They **sum back exactly** to the original signal
- And most intriguingly:
  in an ideal system, they have zero interaction under any inner product

So if, later on, you detect *any* interaction between them, you know one thing immediately:

> something in the system (noise, distortion, misalignment) caused it

This gives you a kind of **self-checking signal decomposition**—no extra redundancy, no pilots, just structure.

That’s exactly what the Walsh–Hadamard operator gives you.

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2. Basic setup

Let $H \in \mathbb{R}^{n \times n}$ be the **normalized Walsh–Hadamard transform**, with:

- $H^T = H$
- $H^2 = I$
- Eigenvalues: $+1, -1$

Define:

$$P_+ = \tfrac{1}{2}(I + H), \quad P_- = \tfrac{1}{2}(I - H)$$

These satisfy:

- $P_+^2 = P_+, P_-^2 = P_-$
- $P_+ P_- = 0$
- $P_+ + P_- = I$

Core identity:

$$(I + H)(I - H) = 0$$

Any signal decomposes as:

$$x = x_+ + x_-, \quad x_+ = P_+ x,\ x_- = P_- x$$

with:

- $x_+ \perp x_-$
- energy splits additively

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3. What makes this special?

Most decompositions separate signals.

This one does something stronger:

> **It guarantees zero cross-interaction by construction.**

So if you ever observe:

$\langle x_+, x_- \rangle \neq 0$

that interaction **cannot come from the signal**—it must come from the system.

This flips the usual perspective:

- Instead of asking “how much noise is there?”
- You ask:
  “how much forbidden interaction has appeared?”

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4. Transmission model and leakage signal

Split and transmit:

$$x_+ = P_+ x, \quad x_- = P_- x$$

Receive:

$$y_+ = x_+ + n_+, \quad y_- = x_- + n_-$$

Define the **leakage statistic**:

$$L = \langle y_+, y_- \rangle$$

Because:

$$\langle x_+, x_- \rangle = 0$$

we get:

$$L = \langle x_+, n_- \rangle + \langle n_+, x_- \rangle + \langle n_+, n_- \rangle$$

Key point:

- L is **zero in the ideal system**
- Any nonzero value is **purely due to imperfections**

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5. Interpreting leakage as channel structure

Let the channel be linear: y =Ax

Then leakage arises from:

$P_+AP_- \neq 0$

So:

> Leakage directly measures **off-diagonal coupling** in the Hadamard eigenbasis

This is richer than scalar noise:

- It captures *how* the channel mixes structured components
- Not just *how much* noise is present

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6. Gaussian signal case (clean baseline)

Let:

$$x \sim \mathcal{N}(0, \sigma^2 I)$$

Then:

- x_+, x_- are independent Gaussian components
- Perfectly orthogonal
- Energy splits evenly

So:

- Expected leakage = 0
- Variance is tightly controlled

This gives:

> A **well-calibrated baseline** where any observed leakage is meaningful

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7. Synchronization and alignment

Orthogonality depends on alignment.

If the system introduces:

- time shifts
- phase errors
- basis mismatch

then:

- x_+ and x_- are no longer orthogonal after the channel
- leakage increases

So you can:

- **minimize leakage** to estimate alignment parameters
- perform **blind synchronization**

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8. Nonlinear effects (where this gets interesting)

If the channel includes:

- clipping
- quantization
- ReLU-like nonlinearities

then:

- orthogonality is broken
- energy leaks across subspaces

So leakage becomes:

> a detector of **nonlinear distortion**, not just additive noise

This is particularly useful because:

- it does not require modeling the nonlinearity

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9. Coding and consistency viewpoint

The decomposition enforces:

$x_+ \perp x_-$

This acts like a **built-in constraint**:

- valid signals lie on a structured manifold
- deviations indicate corruption

Applications:

- error detection via leakage
- projection-based denoising
- consistency checks across channels

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10. System design ideas

- **Dual-channel transmission**
  - send x_+, x_- separately
  - monitor leakage for channel health
- **Implicit monitoring**
  - recompute split at receiver
  - no explicit pilots needed
- **Adaptive equalization**
  - minimize cross-coupling between subspaces
- **Low-overhead diagnostics**
  - leakage is a free byproduct of the representation

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11. Representation learning / neural networks

Replace sign-based splits with:

$$x \mapsto (P_+x, P_-x)$$

Properties:

- lossless
- orthogonal
- balanced

Leakage across layers measures:

- numerical instability
- mixing between structured components
- nonlinear distortion

Potential uses:

- regularization via leakage penalties
- stability diagnostics
- structured residual pathways

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12. Big picture

The identity:

$(I+H)(I-H) = 0$

gives more than a decomposition—it gives a principle:

> **Design representations where valid components are mutually non-interacting. Then any interaction becomes a direct, signal-independent measure of system imperfection.**

This turns structure into a tool for:

- measurement
- diagnostics
- and potentially control